

# Path Integral for Relativistic Three-Dimensional Spinless Aharonov-Bohm-Coulomb System

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## Abstract

The path integral for relativistic three-dimensional spinless Aharonov-Bohm-Coulomb system is solved, and the energy spectra are extracted from the resulting amplitude.

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# 1 INTRODUCTION

In classical mechanics the effect of an electromagnetic field on a charged particle is completely described by means of the Lorentz force equation

$$m \frac{d^2 x^\mu}{d\tau^2} = e F^{\mu\nu} \frac{dx_\nu}{d\tau} \quad (1)$$

where  $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$  is the electromagnetic field tensor and  $d\tau = dt(1 - v^2)^{1/2}$  is the proper time interval. The vector potential  $A^\mu$  is merely an auxiliary field. One could formulate the particle motion in terms of its derivatives only: the electric and magnetic fields.

Surprisingly, it was pointed out in 1959 by AB [1] that, according to quantum mechanics, the motion of a charged particle can be influenced by electromagnetic fields in regions from which the particle is excluded. This is called Aharonov-Bohm (AB) effect and this effect has been well-confirmed experimentally [2]. In the past 15 years, AB effect has been studied in the context of anyonic [3], cosmic string [4], and (2+1)-dimensional gravity theories [5]. Since the anyon, a two dimensional object, carries a magnetic flux [6], the dominant interaction between anyons is the AB interaction. Furthermore, Arovas et al have pointed out in their seminal paper [7] by calculating the second virial coefficient of the anyons, that the thermodynamic behavior of anyon system interpolates between bosons and fermions. An excellent review of past theoretical and experimental work can be found in [9].

Since Feynman propagator is very helpful for analyzing the scattering and statistical property of anyon system, it is important to derive the relativistic propagator of the Aharonov-Bohm-Coulomb (ABC) system. In the past 15 years, considerable progress has been made in solving path integrals of potential problems [10, 11]. However, only few rela-

tivistic problems has been discussed by PI in [10, 12, 13, 14, 16, 15, 17, 18]. In this paper, we solve the relativistic spinless 3-dimensional ABC system by path integral, and the energy spectra are extracted from the resulting amplitude.

## 2 THE RELATIVISTIC PATH INTEGRAL

Adding a vector potential  $\mathbf{A}(\mathbf{x})$  to Kleinert's relativistic path integral for a particle in a potential  $V(\mathbf{x})$  [10, 12], we find that the expression for the fixed-energy amplitude is [15]

$$G(\mathbf{x}_b, \mathbf{x}_a; E) = \frac{i\hbar}{2Mc} \int_0^\infty dL \int D\rho \Phi[\rho] \int D^D x e^{-A_E/\hbar} \quad (2)$$

with the action

$$A_E = \int_{\tau_a}^{\tau_b} d\tau \left[ \frac{M}{2\rho(\tau)} \mathbf{x}'^2(\tau) - i(e/c) \mathbf{A} \cdot \mathbf{x}'(\tau) - \rho(\tau) \frac{(E - V)^2}{2Mc^2} + \rho(\tau) \frac{Mc^2}{2} \right]. \quad (3)$$

For the ABC system under consideration, the potential is

$$V(r) = -e^2/r, \quad (4)$$

and the vector potential

$$A_i = 2g\partial_i\theta, \quad (5)$$

where  $e$  is the charge and  $\theta$  is the azimuthal angle around the tube:

$$\theta(\mathbf{x}) = \arctan(x_2/x_1). \quad (6)$$

The associated magnetic field lines are confined to an infinitely thin tube along the  $z$ -axis:

$$B_3 = 2g\epsilon_{3jk}\partial_j\partial_k\theta = 2g2\pi\delta^{(2)}(\mathbf{x}_\perp), \quad (7)$$

where  $\mathbf{x}_\perp$  is the transverse vector  $\mathbf{x}_\perp \equiv (x_1, x_2)$ .

Before time-slicing the path integral, we have to regularize it via a so-called  $f$ -transformation [10, 13], which exchanges the path parameter  $\tau$  by a new one  $s$ :

$$d\tau = ds f_l(\mathbf{x}_n) f_r(\mathbf{x}_{n-1}), \quad (8)$$

where  $f_l(\mathbf{x})$  and  $f_r(\mathbf{x})$  are invertible functions whose product is positive. The freedom in choosing  $f_{l,r}$  amounts to an invariance under path-dependent-reparametrizations of the path parameter  $\tau$  in the fixed-energy amplitude (2). By this transformation, the (D+1)-dimensional relativistic fixed-energy amplitude for arbitrary time-independent potential turns into [10, 13]

$$\begin{aligned} G(\mathbf{x}_b, \mathbf{x}_a; E) &\approx \frac{i\hbar}{2Mc} \int_0^\infty dS \prod_{n=1}^{N+1} \left[ \int d\rho_n \Phi(\rho_n) \right] \\ &\times \frac{f_l(\mathbf{x}_a) f_r(\mathbf{x}_b)}{\left( \frac{2\pi\hbar\epsilon_b^s \rho_b f_l(\mathbf{x}_b) f_r(\mathbf{x}_a)}{M} \right)^{D/2}} \prod_{n=1}^N \left[ \int_{-\infty}^\infty \frac{d^D x_n}{\left( \frac{2\pi\hbar\epsilon_n^s \rho_n f(\mathbf{x}_n)}{M} \right)^{D/2}} \right] \exp \left\{ -\frac{1}{\hbar} A^N \right\} \end{aligned} \quad (9)$$

with the  $s$ -sliced action

$$\begin{aligned} A^N &= \sum_{n=1}^{N+1} \left[ \frac{M (\mathbf{x}_n - \mathbf{x}_{n-1})^2}{2\epsilon_n^s \rho_n f_l(\mathbf{x}_n) f_r(\mathbf{x}_{n-1})} - i \frac{e}{c} \mathbf{A}_n \cdot (\mathbf{x}_n - \mathbf{x}_{n-1}) \right. \\ &\quad \left. - \epsilon_n^s \rho_n f_l(\mathbf{x}_n) f_r(\mathbf{x}_{n-1}) \frac{(E - V)^2}{2Mc^2} + \epsilon_n^s \rho_n f_l(\mathbf{x}_n) f_r(\mathbf{x}_{n-1}) \frac{Mc^2}{2} \right]. \end{aligned} \quad (10)$$

A family functions which regulates the ABC system is

$$f_l(\mathbf{x}) = f(\mathbf{x})^{1-\lambda}, \quad f_r(\mathbf{x}) = f(\mathbf{x})^\lambda, \quad (11)$$

whose product satisfies  $f_l(\mathbf{x}) f_r(\mathbf{x}) = f(\mathbf{x}) = r$ . Thus arrive at the amplitude

$$\begin{aligned} G(\mathbf{x}_b, \mathbf{x}_a; E) &\approx \frac{i\hbar}{2Mc} \int_0^\infty dS \prod_{n=1}^{N+1} \left[ \int d\rho_n \Phi(\rho_n) \right] \\ &\times \frac{r_a^{1-\lambda} r_b^\lambda}{\left( \frac{2\pi\hbar\epsilon_b^s \rho_b r_b^{1-\lambda} r_a^\lambda}{M} \right)^{3/2}} \prod_{n=2}^{N+1} \left[ \int_{-\infty}^\infty \frac{d^3 \Delta x_n}{\left( \frac{2\pi\hbar\epsilon_n^s \rho_n r_{n-1}}{M} \right)^{3/2}} \right] \exp \left\{ -\frac{1}{\hbar} A^N \right\}, \end{aligned} \quad (12)$$

where the action is

$$A^N = \sum_{n=1}^{N+1} \left[ \frac{M(\mathbf{x}_n - \mathbf{x}_{n-1})^2}{2\epsilon_n^s \rho_n r_n^{1-\lambda} r_{n-1}^\lambda} - i(e/c) \mathbf{A}_n \cdot (\mathbf{x}_n - \mathbf{x}_{n-1}) \right. \\ \left. - \epsilon_n^s \rho_n r_n (r_{n-1}/r_n)^\lambda \frac{(E-V)^2}{2Mc^2} + \epsilon_n^s \rho_n r_n (r_{n-1}/r_n)^\lambda \frac{Mc^2}{2} \right]. \quad (13)$$

For using the Kustaanheimo-Stiefel (KS) transformation (e.g.[10]), we now incorporate the dummy fourth dimension into the action by replacing  $\mathbf{x}$  in the kinetic term by the four-vector  $\vec{x}$  and extending the kinetic action to

$$A_{\text{kin}}^N = \sum_{n=1}^{N+1} \frac{M}{2} \frac{(\vec{x}_n - \vec{x}_{n-1})}{\epsilon_n^s \rho_n r_n^{1-\lambda} r_{n-1}^\lambda}. \quad (14)$$

This is achieved by inserting the following trivial identity

$$\prod_{n=1}^{N+1} \left[ \int \frac{d(\Delta x^4)_n}{(2\pi\hbar\epsilon_n^s \rho_n r_n^{1-\lambda} r_{n-1}^\lambda/M)^{1/2}} \right] \exp \left\{ -\frac{1}{\hbar} \sum_{n=1}^{N+1} \frac{M}{2} \frac{(\Delta x_n^4)^2}{\epsilon_n^s \rho_n r_n^{1-\lambda} r_{n-1}^\lambda} \right\} = 1. \quad (15)$$

Hence the fixed-energy amplitude of the ABC system in three dimensions can be rewritten as the four-dimensional path integral

$$G(\mathbf{x}_b, \mathbf{x}_a; E) \approx \frac{i\hbar}{2Mc} \int_0^\infty dS \prod_{n=1}^{N+1} \left[ \int d\rho_n \Phi(\rho_n) \right] \\ \times \int dx_a^4 \frac{r_a^{1-\lambda} r_b^\lambda}{\left( \frac{2\pi\hbar\epsilon_b^s \rho_b r_b^{1-\lambda} r_a^\lambda}{M} \right)^2} \prod_{n=2}^{N+1} \left[ \int_{-\infty}^\infty \frac{d^4 \Delta x_n}{\left( \frac{2\pi\hbar\epsilon_n^s \rho_n r_{n-1}}{M} \right)^2} \right] \exp \left\{ -\frac{1}{\hbar} A^N \right\}, \quad (16)$$

where  $A^N$  is the action Eq. (13) in which the three-vectors  $\mathbf{x}_n$  are replaced by the four-vectors  $\vec{x}_n$ . With the help of the following approximation

$$\frac{r_a^{1-\lambda} r_b^\lambda}{\left( \frac{2\pi\hbar\epsilon_b^s \rho_b r_b^{1-\lambda} r_a^\lambda}{M} \right)^2} \prod_{n=2}^{N+1} \left[ \int_{-\infty}^\infty \frac{d^4 \Delta x_n}{\left( \frac{2\pi\hbar\epsilon_n^s \rho_n r_{n-1}}{M} \right)^2} \right] \\ \approx \frac{1}{r_a} \frac{1}{\left( \frac{2\pi\hbar\epsilon_b^s \rho_b}{M} \right)^2} \prod_{n=2}^{N+1} \left[ \int_{-\infty}^\infty \frac{d^4 \Delta x_n}{\left( \frac{2\pi\hbar\epsilon_n^s \rho_n r_n}{M} \right)^2} \right] \exp \left\{ 3\lambda \sum_{n=1}^{N+1} \log \frac{r_n}{r_{n-1}} \right\}, \quad (17)$$

where the equality  $(r_b/r_a)^{3\lambda-2} = \prod_1^{N+1} (r_n/r_{n-1})^{3\lambda-2}$  has been used, we arrive

$$G(\mathbf{x}_b, \mathbf{x}_a; E) \approx \frac{i\hbar}{2Mc} \int_0^\infty dS \prod_{n=1}^{N+1} \left[ \int d\rho_n \Phi(\rho_n) \right] \\ \times \int dx_a^4 \frac{1}{r_a} \frac{1}{\left( \frac{2\pi\hbar e_b^s \rho_b}{M} \right)^2} \prod_{n=2}^{N+1} \left[ \int_{-\infty}^\infty \frac{d^4 \Delta x_n}{\left( \frac{2\pi\hbar e_n^s \rho_n r_n}{M} \right)^2} \right] \exp \left\{ -\frac{1}{\hbar} \sum_{n=1}^{N+1} \left[ A^N - 3\lambda\hbar \log \frac{r_n}{r_{n-1}} \right] \right\}. \quad (18)$$

Since the path integral represents the general relativistic resolvent operator, all results must be independent of the splitting parameter  $\lambda$  after going to the continuum limit. Choosing a splitting parameter  $\lambda = 0$ , we obtain the continuum limit of the action

$$A_E [x, x'] = \int_0^S ds \left[ \frac{Mx'^2}{2\rho r} - i(e/c)\mathbf{A} \cdot \mathbf{x}' - \rho r \frac{(E-V)^2}{2Mc^2} + \rho r \frac{Mc^2}{2} \right] \quad (19)$$

We now solve the ABC system by introducing the KS transformation (e.g.[10])

$$d\vec{x} = 2A(\vec{u})d\vec{u}. \quad (20)$$

The arrow on top of  $x$  indicates that  $x$  has become a four-vector. For symmetry reasons, we choose the  $4 \times 4$  matrix  $A(\vec{u})$  as

$$A(\vec{u}) = \begin{pmatrix} u^3 & u^4 & u^1 & u^2 \\ u^4 & -u^3 & -u^2 & u^1 \\ u^1 & u^2 & -u^3 & -u^4 \\ u^2 & -u^1 & u^4 & -u^3 \end{pmatrix}. \quad (21)$$

The transformation of coordinate difference is

$$(\Delta \mathbf{x}_n^i)^2 = 4\bar{\mathbf{u}}_n^2 (\Delta \mathbf{u}_n^i)^2, \quad (22)$$

where  $\bar{\mathbf{u}}_n \equiv (\mathbf{u}_n + \mathbf{u}_{n-1})/2$ . In the continuum limit, this amounts to

$$d^4 x_n = 16\mathbf{u}_n^2 d^4 u_n = 16r_n^2 d^4 u_n, \quad (23)$$

$$\vec{x}'^2 = 4\bar{u}^2 \vec{u}'^2 = 4r\bar{u}'^2. \quad (24)$$

By employing the basis tetrad notation  $e^i{}_\mu(\vec{u})$ , Eq. (20) has the form  $dx^i = e^i{}_\mu(\vec{u}) du^\mu$ , this is given by

$$e^i{}_\mu(\vec{u}) = \frac{\partial x^i}{\partial u^\mu}(\vec{u}) = 2A^i{}_\mu(\vec{u}), \quad i = 1, 2, 3, 4. \quad (25)$$

Under the KS transformation, the magnetic interaction turns into

$$\begin{aligned} \mathbf{A}_n \cdot (\mathbf{x}_n - \mathbf{x}_{n-1}) &= -2g \frac{x_n^2 \Delta x_n^1 - x_n^1 \Delta x_n^2}{r_n^2} \\ &= -2g \left[ \frac{u_n^1 \Delta u_n^2 - u_n^2 \Delta u_n^1}{(u_n^1)^2 + (u_n^2)^2} + \frac{u_n^4 \Delta u_n^3 - u_n^3 \Delta u_n^4}{(u_n^3)^2 + (u_n^4)^2} \right]. \end{aligned} \quad (26)$$

We obtain a path integral equivalent to Eq. (16)

$$G(\mathbf{x}_b, \mathbf{x}_a; E) = \frac{i\hbar}{2Mc} \int_0^\infty dS e^{SEe^2/\hbar Mc^2} G(\vec{u}_b, \vec{u}_a; S), \quad (27)$$

where  $G(\vec{u}_b, \vec{u}_a; S)$  denotes the s-sliced amplitude

$$\prod_{n=1}^{N+1} \left[ \int d\rho_n \Phi(\rho_n) \right] \frac{1}{16} \int \frac{dx_a^4}{r_a} \frac{1}{\left( \frac{2\pi\hbar\epsilon_b^s \rho_b}{m} \right)^2} \prod_{n=1}^N \left[ \int_{-\infty}^\infty \frac{d^4 u_n}{\left( \frac{2\pi\hbar\epsilon_n^s \rho_n}{m} \right)^2} \right] \exp \left\{ -\frac{1}{\hbar} A^N \right\} \quad (28)$$

with the action

$$A^N = \sum_{n=1}^{N+1} \left\{ \frac{m(\Delta \vec{u}_n)^2}{2\epsilon_n^s \rho_n} - i(e/c)(\vec{A}_n \cdot \Delta \vec{u}_n) + \epsilon_n^s \rho_n \frac{m\omega^2 \vec{u}_n^2}{2} - \epsilon_n^s \rho_n \frac{\hbar^2 4\alpha^2}{2m\vec{u}_n^2} \right\}. \quad (29)$$

Here

$$m = 4M, \quad \omega^2 = \frac{M^2 c^4 - E^2}{4M^2 c^2}, \quad (30)$$

and

$$\vec{A}_n \cdot \Delta \vec{u}_n = -2g \left[ \frac{u_n^1 \Delta u_n^2 - u_n^2 \Delta u_n^1}{(u_n^1)^2 + (u_n^2)^2} + \frac{u_n^4 \Delta u_n^3 - u_n^3 \Delta u_n^4}{(u_n^3)^2 + (u_n^4)^2} \right]. \quad (31)$$

We now choose the gauge  $\rho(s) = 1$  in Eq. (28). This leads to the Duru-Kleinert transformed action

$$A = \int_0^S ds \left[ \frac{m\vec{u}'^2}{2} - 2i(e/c)(\vec{A} \cdot \vec{u}') + \frac{m\omega^2 \vec{u}^2}{2} - \frac{4\hbar^2 \alpha^2}{2m\vec{u}^2} \right]. \quad (32)$$

It describes a particle, forgetting the magnetic interaction term for a while, of mass  $m = 4M$  moving as a function of the “pseudotime”  $s$  in a 4-dimensional harmonic oscillator potential of frequency

$$\omega^2 = \frac{M^2 c^4 - E^2}{4M^2 c^2}. \quad (33)$$

The oscillator possesses an additional attractive potential  $-4\hbar^2\alpha^2/2m\vec{u}^2$  which is conveniently parametrized in the form of a centrifugal barrier

$$V_{\text{extra}} = \hbar^2 \frac{l_{\text{extra}}^2}{2m\vec{u}^2}, \quad (34)$$

whose squared angular momentum has the negative value  $l_{\text{extra}}^2 \equiv -4\alpha^2$ , where  $\alpha$  denotes the fine-structure constant  $\alpha \equiv e^2/\hbar c \approx 1/137$ .

There are no  $\lambda$ -slicing corrections. This is ensured by the affine connection of KS transformation satisfying

$$\Gamma_\mu^{\mu\lambda} = g^{\mu\nu} e_i^\lambda \partial_\mu e^i_\nu = 0 \quad (35)$$

and the transverse gauge  $\partial_\mu A^\mu = 0$  [10, 12]. We now analyze the effect come from the magnetic interaction. Note that the system separable like  $R^4 \rightarrow R^2 \times R^2$  if the centrifugal term is not considered for a while. Therefore the path integral in  $u$  space become two independent two-dimensional AB plus harmonic oscillator. This makes the path integral calculation of  $G(\vec{u}_b, \vec{u}_a; S)$  extremely simple. For each two-dimensional system, the derivatives in front of  $\varphi$  in Eq. (7) commute everywhere, except at the origin where Stokes’ theorem yields

$$\int d^2 u (\partial_1 \partial_2 - \partial_2 \partial_1) \varphi = \oint d\varphi = 2\pi \quad (36)$$

The magnetic flux through the tube is defined by the integral

$$\Phi = \int d^2 u B_3. \quad (37)$$

A comparison with Eq. (7) shows that the coupling constant in Eq. (5) is related to the magnetic flux by

$$g = \frac{\Phi}{4\pi}. \quad (38)$$

When inserting  $A_i = 2g\partial_i\varphi$  into Eq. (32), the interaction takes the form

$$A_{\text{mag}} = -\hbar\beta_0 \int_0^S ds\varphi', \quad (39)$$

where  $\beta_0$  is the dimensionless number

$$\beta_0 \equiv -\frac{2eg}{\hbar c}. \quad (40)$$

The minus sign is a matter of convention. Since the particle orbits are present at all times, their worldlines in spacetime can be considered as being closed at infinity, and the integral

$$n = \frac{1}{2\pi} \int_0^S ds\varphi' \quad (41)$$

is the topological invariant with integer values of the winding number  $n$ . The magnetic interaction is therefore a purely topological one, its value being

$$A_{\text{mag}} = -\hbar\beta_0 2\pi n. \quad (42)$$

After adding this to the action of Eq. (32) in the radial decomposition of the relativistic path integral [10, 13], we rewrite the sum over the azimuthal quantum numbers  $m$  via Poisson's summation formula, and obtain

$$\begin{aligned} G(\mathbf{u}_b, \mathbf{u}_a; S) &= \int_{-\infty}^{\infty} d\beta \frac{1}{\sqrt{u_b u_a}} G(u_b, u_a; S)_\beta \\ &\times \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} e^{i(\beta - \beta_0)(\varphi_b + 2n\pi - \varphi_a)}. \end{aligned} \quad (43)$$

Since the winding number  $n$  is often not easy to measure experimentally, let us extract observable consequences which are independent of  $n$ . The sum over all  $n$  forces  $\beta$  to be equal to  $\beta_0$  modula an arbitrary integer number. the result, for each  $R^2$ , is

$$G(\mathbf{u}_b, \mathbf{u}_a; S) = \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{u_b u_a}} G(u_b, u_a; S)_{k+\beta_0} \frac{1}{2\pi} e^{ik(\varphi_b - \varphi_a)}. \quad (44)$$

Therefore we obtain the fixed-energy amplitude

$$\begin{aligned} G(\mathbf{x}_b, \mathbf{x}_a; E) &= \frac{i\hbar}{2Mc} \int_0^\infty dS e^{SEe^2/\hbar Mc^2} \frac{1}{16} \int \frac{dx_a^4}{r_a} \left( \frac{m\omega}{\hbar \sinh \omega s} \right)^2 \\ &\times \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} e^{ik_1(\varphi_{1,b} - \varphi_{1,a})} e^{ik_1(\varphi_{2,b} - \varphi_{2,a})} \\ &\times \exp \left\{ -\frac{m\omega}{2\hbar} (\sigma_{1,b}^2 + \sigma_{1,a}^2 + \sigma_{2,b}^2 + \sigma_{2,a}^2) \coth \omega s \right\} \\ &\times I_{|k_1+\beta_0|} \left( \frac{m}{\hbar} \frac{\omega \sigma_{1,b} \sigma_{1,a}}{\sinh \omega s} \right) I_{|k_2+\beta_0|} \left( \frac{m}{\hbar} \frac{\omega \sigma_{2,b} \sigma_{2,a}}{\sinh \omega s} \right), \end{aligned} \quad (45)$$

where  $(\sigma_1, \varphi_1)$  and  $(\sigma_2, \varphi_2)$  are defined by

$$\left. \begin{array}{l} u^1 = \sigma_1 \sin \varphi_1 \\ u^2 = \sigma_1 \cos \varphi_1 \\ u^3 = \sigma_2 \cos \varphi_2 \\ u^4 = \sigma_2 \sin \varphi_2 \end{array} \right\}. \quad (46)$$

In order to perform the  $x_a^4$ -integration we express  $(\sigma_1, \varphi_1, \sigma_2, \varphi_2)$  in terms of three-dimensional spherical coordinate with an auxiliary angle  $\gamma$ :

$$\left. \begin{array}{l} u^1 = \sqrt{r} \cos(\theta/2) \cos [(\varphi + \gamma)/2] \\ u^2 = \sqrt{r} \cos(\theta/2) \sin [(\varphi + \gamma)/2] \\ u^3 = \sqrt{r} \sin(\theta/2) \cos [(\varphi - \gamma)/2] \\ u^4 = \sqrt{r} \sin(\theta/2) \sin [(\varphi - \gamma)/2] \end{array} \right\} \quad \left( \begin{array}{l} 0 \leq \theta \leq \pi \\ 0 \leq \varphi \leq 2\pi \\ 0 \leq \gamma \leq 4\pi \end{array} \right) \quad (47)$$

and identify

$$\left. \begin{array}{l} \sigma_1 = \sqrt{r} \cos(\theta/2) \\ \varphi_1 = (\varphi + \gamma + \pi)/2 \\ \sigma_2 = \sqrt{r} \sin(\theta/2) \\ \varphi_2 = (\varphi - \gamma)/2 \end{array} \right\}. \quad (48)$$

Then one can change the  $x_a^4$ -integration into the  $\gamma_a$ -integration whose result is easily represented as the Kronecker delta  $\delta_{k_1, k_2}$ . Hence, one can carry out  $k_2$ -summation and finally becomes

$$\begin{aligned} G(\mathbf{x}_b, \mathbf{x}_a; E) &= \frac{i\hbar}{2Mc} \frac{M^2\omega}{\pi\hbar^2} \sum_{k=-\infty}^{\infty} e^{ik(\varphi_b - \varphi_a)} \\ &\times \int_0^\infty dy e^{2(Ee^2/2\omega\hbar Mc^2)y} \frac{1}{\sinh^2 y} e^{-\frac{m\omega}{2\hbar}(r_b + r_a) \coth y} \\ &\times I_{|k+\beta_0|} \left( \frac{m\omega\sqrt{r_b r_a}}{\hbar \sinh y} \cos \theta_b/2 \cos \theta_a/2 \right) I_{|k+\beta_0|} \left( \frac{m\omega\sqrt{r_b r_a}}{\hbar \sinh y} \sin \theta_b/2 \sin \theta_a/2 \right), \end{aligned} \quad (49)$$

where we have defined the new variable  $y = \omega s$ . We make now use of the addition theorem Ref. [19], Vol. II, p.99:

$$\begin{aligned} &\frac{z}{2} J_\nu(z \sin \alpha \sin \beta) J_\mu(z \cos \alpha \cos \beta) \\ &= (\sin \alpha \sin \beta)^\nu (\cos \alpha \cos \beta)^\mu \sum_{l=0}^{\infty} (-1)^l (\mu + \nu + 2l + 1) \\ &\times \frac{\Gamma(\mu + \nu + l + 1) \Gamma(\nu + l + 1)}{l! \Gamma(\mu + l + 1) \Gamma^2(\nu + 1)} J_{\mu+\nu+l+1}(z) \\ &\times {}_2F_1(-l, \mu + \nu + l + 1, \nu + 1; \sin^2 \alpha) \\ &\times {}_2F_1(-l, \mu + \nu + l + 1, \nu + 1; \sin^2 \beta), \end{aligned} \quad (50)$$

and the relation between hypergeometric function and Jacobi-polynomials

$$P_l^{(\alpha, \beta)}(z) = \frac{\Gamma(\alpha + l + 1)}{\Gamma(\alpha + 1) l!} {}_2F_1\left(\alpha + \beta + l + 1, -l; 1 + \alpha; \frac{1-z}{2}\right). \quad (52)$$

We arrive

$$\begin{aligned} G(\mathbf{x}_b, \mathbf{x}_a; E) &= \frac{i\hbar}{2Mc} \frac{M}{2\pi\hbar\sqrt{r_b r_a}} \sum_{k=-\infty}^{\infty} \sum_{n=0}^{\infty} e^{ik(\varphi_b - \varphi_a)} \\ &\times (\cos \theta_b/2 \cos \theta_a/2)^{|k+\beta_0|} (\sin \theta_b/2 \sin \theta_a/2)^{|k+\beta_0|} \\ &\times \frac{n! \Gamma(n+2 | k+\beta_0 | + 1) (2n+2 | k+\beta_0 | + 1)}{\Gamma^2(n+2 | k+\beta_0 | + 1)} \end{aligned}$$

$$\begin{aligned} & \times \left\{ \int_0^\infty dy e^{2(Ee^2/2\omega\hbar Mc^2)y} \frac{1}{\sinh y} e^{-\frac{m\omega}{2\hbar}(r_b+r_a)\coth y} I_{2n+2|k+\beta_0|+1} \left( \frac{m\omega\sqrt{r_b r_a}}{\hbar \sinh y} \right) \right\} \\ & \times P_n^{(|k+\beta_0|, |k+\beta_0|)}(\cos \theta_b) P_n^{(|k+\beta_0|, |k+\beta_0|)}(\cos \theta_a). \end{aligned} \quad (53)$$

At this place, the additional centrifugal barrier (34) is incorporated via the replacement [10]

$$(2n+2 \mid k+\beta_0 \mid +1) \rightarrow \sqrt{(2n+2 \mid k+\beta_0 \mid +1)^2 - 4\alpha^2}. \quad (54)$$

This integral can be calculated by employing the formula

$$\begin{aligned} & \int_0^\infty dy \frac{e^{2\nu y}}{\sinh y} \exp \left[ -\frac{t}{2} (\zeta_a + \zeta_b) \coth y \right] I_\mu \left( \frac{t\sqrt{\zeta_b \zeta_a}}{\sinh y} \right) \\ & = \frac{\Gamma((1+\mu)/2 - \nu)}{t\sqrt{\zeta_b \zeta_a} \Gamma(\mu+1)} W_{\nu, \mu/2}(t\zeta_b) M_{\nu, \mu/2}(t\zeta_b), \end{aligned} \quad (55)$$

with the range of validity

$$\begin{aligned} & \zeta_b > \zeta_a > 0, \\ & \text{Re}[(1+\mu)/2 - \nu] > 0, \\ & \text{Re}(t) > 0, |\arg t| < \pi, \end{aligned}$$

where  $M_{\mu, \nu}$  and  $W_{\mu, \nu}$  are the Whittaker functions, we complete the integration of Eq. (56),

and find the amplitude for  $u_b > u_a$  in the closed form,

$$\begin{aligned} G(\mathbf{x}_b, \mathbf{x}_a; E) &= \frac{i\hbar}{2Mc} \frac{Mc}{4\pi r_b r_a \sqrt{M^2 c^4 - E^2}} \sum_{k=-\infty}^{\infty} \sum_{n=0}^{\infty} e^{ik(\varphi_b - \varphi_a)} \\ & \times (\cos \theta_b/2 \cos \theta_a/2)^{|k+\beta_0|} (\sin \theta_b/2 \sin \theta_a/2)^{|k+\beta_0|} \\ & \times \frac{n! \Gamma(n+2 \mid k+\beta_0 \mid +1) (2n+2 \mid k+\beta_0 \mid +1)}{\Gamma^2(n+2 \mid k+\beta_0 \mid +1)} \\ & \times \frac{\Gamma\left(\frac{1}{2} + \frac{1}{2}\sqrt{(2n+2 \mid k+\beta_0 \mid +1)^2 - 4\alpha^2} - \frac{E\alpha}{\sqrt{M^2 c^4 - E^2}}\right)}{\Gamma\left(\sqrt{(2n+2 \mid k+\beta_0 \mid +1)^2 - 4\alpha^2} + 1\right)} \\ & \times W_{E\alpha/\sqrt{M^2 c^4 - E^2}, \sqrt{(2n+2 \mid k+\beta_0 \mid +1)^2 - 4\alpha^2}/2} \left( \frac{2}{\hbar c} \sqrt{M^2 c^4 - E^2} r_b \right) \\ & \times M_{E\alpha/\sqrt{M^2 c^4 - E^2}, \sqrt{(2n+2 \mid k+\beta_0 \mid +1)^2 - 4\alpha^2}/2} \left( \frac{2}{\hbar c} \sqrt{M^2 c^4 - E^2} r_a \right) \\ & \times P_n^{(|k+\beta_0|, |k+\beta_0|)}(\cos \theta_b) P_n^{(|k+\beta_0|, |k+\beta_0|)}(\cos \theta_a). \end{aligned} \quad (56)$$

The energy spectra can be extracted from the poles. They are determined by

$$\frac{1}{2} + \frac{1}{2}\sqrt{(2n+2|k+\beta_0|+1)^2 - 4\alpha^2} - \frac{E\alpha}{\sqrt{M^2c^4 - E^2}} = -n_r, \quad n_r = 0, 1, 2, \dots \quad (57)$$

Expanding this equation into a power of  $\alpha$ , we get

$$E_{n_r,n,k} = \pm Mc^2 \left\{ 1 - \frac{1}{2} \left( \frac{\alpha}{n_r + n + |k + \beta_0| + 1} \right)^2 - \frac{\alpha^4}{(n_r + n + |k + \beta_0| + 1)^3} \right. \\ \times \left[ \frac{1}{2n+2|k+\beta_0|+1} - \frac{3}{8(n_r+n+|k+\beta_0|+1)} \right] + O(\alpha^6) \Bigg\}, \quad n_r, n = 0, 1, 2, 3, \dots \quad (58)$$

In the non-relativistic limit, the spectra is in agreement with the result in Ref. [8, 20, 21].

The alert reader will have noted the similarity of the techniques used in this paper to those leading to the solution of the path integral of the dionium atom [22].

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